

Exact penalty decomposition method for zero-norm minimization based on MPEC formulation ¹

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Abstract. We reformulate the zero-norm minimization problem as an equivalent mathematical program with equilibrium constraints and establish that its penalty problem, induced by adding the complementarity constraint to the objective, is exact. Then, by the special structure of the exact penalty problem, we propose a decomposition method that can seek a global optimal solution of the zero-norm minimization problem under the null space condition in [23] by solving a finite number of weighted l_1 -norm minimization problems. To handle the weighted l_1 -norm subproblems, we develop a partial proximal point algorithm where the subproblems may be solved approximately with the limited memory BFGS (L-BFGS) or the semismooth Newton-CG. Finally, we apply the exact penalty decomposition method with the weighted l_1 -norm subproblems solved by combining the L-BFGS with the semismooth Newton-CG to several types of sparse optimization problems, and compare its performance with that of the penalty decomposition method [25], the iterative support detection method [38] and the state-of-the-art code **FPC_AS** [39]. Numerical comparisons indicate that the proposed method is very efficient in terms of the recoverability and the required computing time.

Key words: zero-norm minimization; MPECs; exact penalty; decomposition method.

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1 Introduction

Let \mathbb{R}^n be the real vector space of dimension n endowed with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. We consider the following zero-norm minimization problem

$$\min_{x \in \mathbb{R}^n} \left\{ \|x\|_0 : \|Ax - b\| \leq \delta \right\}, \quad (1)$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given data, $\delta \geq 0$ is a given constant, and

$$\|x\|_0 := \sum_{i=1}^n \text{card}(x_i) \quad \text{with} \quad \text{card}(x_i) = \begin{cases} 1 & \text{if } x_i \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this paper, we denote by \mathcal{F} the feasible set of problem (1) and assume that it is nonempty. This implies that (1) has a nonempty set of globally optimal solutions.

The problem (1) has very wide applications in sparse reconstruction of signals and images (see, e.g., [14, 6, 10, 11]), sparse model selection (see, e.g., [36, 15]) and error correction [8]. For example, when considering the recovery of signals from noisy data, one may solve (1) with some given $\delta > 0$. However, due to the discontinuity and nonconvexity of zero-norm, it is difficult to find a globally optimal solution of (1). In addition, each feasible solution of (1) is locally optimal, but the number of its globally optimal solutions is finite when $\delta = 0$, which brings in more difficulty to its solving. A common way is to obtain a favorable locally optimal solution by solving a convex surrogate problem such as the l_1 -norm minimization or l_1 -norm regularized problem. In the past two decades, this convex relaxation technique became very popular due to the important results obtained in [13, 14, 36, 12]. Among others, the results of [13, 14] quantify the ability of l_1 -norm minimization problem to recover sparse reflectivity functions. For brief historical accounts on the use of l_1 -norm minimization in statistics and signal processing, please see [29, 37]. Motivated by this, many algorithms have been proposed for the l_1 -norm minimization or l_1 -norm regularized problems (see, e.g., [9, 17, 24]).

Observing the key difference between the l_1 -norm and the l_0 -norm, Candès et al. [8] recently proposed a reweighted l_1 -norm minimization method (the idea of this method is due to Fazel [16] where she first applied it for the matrix rank minimization). This method is solving a sequence of convex relaxations of the following nonconvex surrogate

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{i=1}^n \ln(|x_i| + \varepsilon) : \|Ax - b\| \leq \delta \right\}. \quad (2)$$

This class of surrogate problems are further studied in [32, 41]. In addition, noting that the l_p -norm $\|x\|_p^p$ tends to $\|x\|_0$ as $p \rightarrow 0$, many researchers seek a locally optimal solution of the problem (1) by solving the nonconvex approximation problem

$$\min_{x \in \mathbb{R}^n} \left\{ \|x\|_p^p : \|Ax - b\| \leq \delta \right\}$$

or its regularized formulation (see, e.g., [18, 10]). Extensive computational studies in [8, 18, 10, 32] demonstrate that the reweighted l_1 -norm minimization method and the l_p -norm nonconvex approximation method can find sparser solutions than the l_1 -norm convex relaxation method. We see that all the methods mentioned above are developed by the surrogates or the approximation of zero-norm minimization problem.

In this paper, we reformulate (1) as an equivalent MPEC (mathematical program with equilibrium constraints) by the variational characterization of zero-norm, and then establish that its penalty problem, induced by adding the complementarity constraint to the objective, is exact, i.e., the set of globally optimal solutions of the penalty problem coincides with that of (1) when the penalty parameter is over some threshold. Though the exact penalty problem itself is also difficult to solve, we exploit its special structure to propose a decomposition method that is actually a reweighted l_1 -norm minimization method. This method, consisting of a finite number of weighted l_1 -norm minimization, is shown to yield a favorable locally optimal solution, and moreover a globally optimal solution of (1) under the null space condition in [23]. For the weighted l_1 -norm minimization problems, there are many softwares suitable for solving them such as the alternating direction method software YALL1 [42], and we here propose a partial proximal point method where the proximal point subproblems may be solved approximately with the L-BFGS or the semismooth Newton-CG method (see Section 4).

We test the performance of the exact penalty decomposition method with the subproblems solved by combining the L-BFGS and the semismooth Newton-CG for the problems with several types of sparsity, and compare its performance with that of the penalty decomposition method (**QPDM**) [25], the iterative support detection method (**ISDM**) [38] and the state-of-the-art code **FPC_AS** [39]. Numerical comparisons show that the proposed method has a very good robustness, can find the sparsest solution with desired feasibility for the Sparco collection, has comparable recoverability with **ISDM** from fewer observations for most of randomly generated problems which is higher than that of **FPC_AS** and **QPDM**, and requires less computing time than **ISDM**.

Notice that the **ISDM** proposed in [38] is also a reweighted l_1 -norm minimization method in which, the weight vector involved in each weighted l_1 minimization problem is chosen as the support of some index set. Our exact penalty decomposition method shares this feature with **ISDM**, but the index sets to determine the weight vectors are automatically yielded by relaxing the exact penalty problem of the MPEC equivalent to the zero-norm problem (1), instead of using some heuristic strategy. In particular, our theoretical analysis for the exact recovery is based on the null space condition in [23] which is weaker than the truncated null space condition in [38]. Also, the weighted l_1 -norm subproblems involved in our method are solved by combining the L-BFGS with the semismooth Newton-CG method, while such problems in [38] are solved by applying YALL1 [42] directly. Numerical comparisons show that the hybrid of the L-BFGS with the semismooth Newton-CG method is effective for handling the weighted l_1 -norm

subproblems. The penalty decomposition method proposed by Lu and Zhang [25] aims to deal with the zero-norm minimization problem (1), but their method is based on a quadratic penalty for the equivalent augmented formulation of (1), and numerical comparisons show that such penalty decomposition method has a very worse recoverability than **FPC_AS** which is designed for solving the l_1 -minimization problem.

Unless otherwise stated, in the sequel, we denote by e a column vector of all 1s whose dimension is known from the context. For any $x \in \mathbb{R}^n$, $\text{sign}(x)$ denotes the sign vector of x , x^\downarrow is the vector of components of x being arranged in the nonincreasing order, and x_I denotes the subvector of components whose indices belong to $I \subseteq \{1, \dots, n\}$. For any matrix $A \in \mathbb{R}^{m \times n}$, we write $\text{Null}(A)$ as the null space of A . Given a point $x \in \mathbb{R}^n$ and a constant $\gamma > 0$, we denote by $\mathcal{N}(x, \gamma)$ the γ -open neighborhood centered at x .

2 Equivalent MPEC formulation

In this section, we provide the variational characterization of zero-norm and reformulate (1) as a MPEC problem. For any given $x \in \mathbb{R}^n$, it is not hard to verify that

$$\|x\|_0 = \min_{v \in \mathbb{R}^n} \left\{ \langle e, e - v \rangle : \langle v, |x| \rangle = 0, 0 \leq v \leq e \right\}. \quad (3)$$

This implies that the zero-norm minimization problem (1) is equivalent to

$$\min_{x, v \in \mathbb{R}^n} \left\{ \langle e, e - v \rangle : \|Ax - b\| \leq \delta, \langle v, |x| \rangle = 0, 0 \leq v \leq e \right\}, \quad (4)$$

which is a mathematical programming problem with the complementarity constraint:

$$\langle v, |x| \rangle = 0, \quad v \geq 0, \quad |x| \geq 0.$$

Notice that the minimization problem on the right hand side of (3) has a unique optimal solution $v^* = e - \text{sign}(|x|)$, although it is only a convex programming problem. Such a variational characterization of zero-norm was given in [1] and [21, Section 2.5], but there it is not used to develop any algorithms for the zero-norm problems. In the next section, we will develop an algorithm for (1) based on an exact penalty formulation of (4).

Though (4) is given by expanding the original problem (1), the following proposition shows that such expansion does not increase the number of locally optimal solutions.

Proposition 2.1 *For problems (1) and (4), the following statements hold.*

- (a) *Each locally optimal solution of (4) has the form $(x^*, e - \text{sign}(|x^*|))$.*
- (b) *x^* is locally optimal to (1) if and only if $(x^*, e - \text{sign}(|x^*|))$ is locally optimal to (4).*

Proof. (a) Let (x^*, v^*) be an arbitrary locally optimal solution of (4). Then there is an open neighborhood $\mathcal{N}((x^*, v^*), \gamma)$ such that $\langle e, e - v \rangle \geq \langle e, e - v^* \rangle$ for all $(x, v) \in \mathcal{S} \cap \mathcal{N}((x^*, v^*), \gamma)$, where \mathcal{S} is the feasible set of (4). Consider (3) associated to x^* , i.e.,

$$\min_{v \in \mathbb{R}^n} \left\{ \langle e, e - v \rangle : \langle v, |x^*| \rangle = 0, 0 \leq v \leq e \right\}. \quad (5)$$

Let $v \in \mathbb{R}^n$ be an arbitrary feasible solution of problem (5) and satisfy $\|v - v^*\| \leq \gamma$. Then, it is easy to see that $(x^*, v) \in \mathcal{S} \cap \mathcal{N}((x^*, v^*), \gamma)$, and so $\langle e, e - v \rangle \geq \langle e, e - v^* \rangle$. This shows that v^* is a locally optimal solution of (5). Since (5) is a convex optimization problem, v^* is also a globally optimal solution. However, $e - \text{sign}(|x^*|)$ is the unique optimal solution of (5). This implies that $v^* = e - \text{sign}(|x^*|)$.

(b) Assume that x^* is a locally optimal solution of (1). Then, there exists an open neighborhood $\mathcal{N}(x^*, \gamma)$ such that $\|x\|_0 \geq \|x^*\|_0$ for all $x \in \mathcal{F} \cap \mathcal{N}(x^*, \gamma)$. Let (x, v) be an arbitrary feasible point of (4) such that $\|(x, v) - (x^*, e - \text{sign}(|x^*|))\| \leq \gamma$. Then, since v is a feasible point of the problem on the right hand side of (3), we have

$$\langle e, e - v \rangle \geq \|x\|_0 \geq \|x^*\|_0 = \langle e, e - \text{sign}(|x^*|) \rangle.$$

This shows that $(x^*, e - \text{sign}(|x^*|))$ is a locally optimal solution of (4).

Conversely, assume that $(x^*, e - \text{sign}(|x^*|))$ is a locally optimal solution of (4). Then, for any sufficiently small $\gamma > 0$, it clearly holds that $\|x\|_0 \geq \|x^*\|_0$ for all $x \in \mathcal{N}(x^*, \gamma)$. This means that for any $x \in \mathcal{F} \cap \mathcal{N}(x^*, \gamma)$, we have $\|x\|_0 \geq \|x^*\|_0$. Hence, x^* is a locally optimal solution of (1). The two sides complete the proof of part (b). \square

3 Exact penalty decomposition method

In the last section we established the equivalence of (1) and (4). In this section we show that solving (4), and then solving (1), is equivalent to solving a single penalty problem

$$\min_{x, v \in \mathbb{R}^n} \left\{ \langle e, e - v \rangle + \rho \langle v, |x| \rangle : \|Ax - b\| \leq \delta, 0 \leq v \leq e \right\} \quad (6)$$

where $\rho > 0$ is the penalty parameter, and then develop a decomposition method for (4) based on this penalty problem. It is worthwhile to point out that there are many papers studying exact penalty for bilevel linear programs or general MPECs (see, e.g., [4, 5, 28, 26]), but these references do not imply the exactness of penalty problem (6).

For convenience, we denote by \mathcal{S} and \mathcal{S}^* the feasible set and the optimal solution set of problem (4), respectively; and for any given $\rho > 0$, denote by \mathcal{S}_ρ and \mathcal{S}_ρ^* the feasible set and the optimal solution set of problem (6), respectively.

Lemma 3.1 *For any given $\rho > 0$, the problem (6) has a nonempty optimal solution set.*

Proof. Notice that the objective function of problem (6) has a lower bound in \mathcal{S}_ρ , to say α^* . Therefore, there must exist a sequence $\{(x^k, v^k)\} \subset \mathcal{S}_\rho$ such that for each k ,

$$\langle e, e - v^k \rangle + \rho \langle |x^k|, v^k \rangle \leq \alpha^* + \frac{1}{k}. \quad (7)$$

Since the sequence $\{v^k\}$ is bounded, if the sequence $\{x^k\}$ is also bounded, then by letting (\bar{x}, \bar{v}) be an arbitrary limit point of $\{(x^k, v^k)\}$, we have $(\bar{x}, \bar{v}) \in \mathcal{S}_\rho$ and

$$\langle e, e - \bar{v} \rangle + \rho \langle |\bar{x}|, \bar{v} \rangle \leq \alpha^*,$$

which implies that (\bar{x}, \bar{v}) is a globally optimal solution of (6). We next consider the case where the sequence $\{x^k\}$ is unbounded. Define the disjoint index sets I and \bar{I} by

$$I := \{i \in \{1, \dots, n\} \mid \{x_i^k\} \text{ is unbounded}\} \quad \text{and} \quad \bar{I} := \{1, \dots, n\} \setminus I.$$

Since $\{v^k\}$ is bounded, we without loss of generality assume that it converges to \bar{v} . From equation (7), it then follows that $\bar{v}_I = 0$. Note that the sequence $\{x^k\} \subset \mathbb{R}^n$ satisfies $A_I x_I^k + A_{\bar{I}} x_{\bar{I}}^k = b + \Delta^k$ with $\|\Delta^k\| \leq \delta$. Since the sequences $\{x_{\bar{I}}^k\}$ and $\{\Delta^k\}$ are bounded, we may assume that they converge to $\bar{x}_{\bar{I}}$ and Δ , respectively. Then, from the closedness of the set $A_I \mathbb{R}^I$, there exists an $\xi \in \mathbb{R}^I$ such that $A_I \xi + A_{\bar{I}} \bar{x}_{\bar{I}} = b + \Delta$ with $\|\Delta\| \leq \delta$. Letting $\bar{x} = (\xi, \bar{x}_{\bar{I}}) \in \mathbb{R}^n$, we have $(\bar{x}, \bar{v}) \in \mathcal{S}_\rho$. Moreover, the following inequalities hold:

$$\begin{aligned} \langle e, e - \bar{v} \rangle + \rho \langle |\bar{x}|, \bar{v} \rangle &= \langle e, e - \bar{v} \rangle + \rho \sum_{i \in \bar{I}} |\bar{x}_i| \bar{v}_i = \lim_{k \rightarrow \infty} \left[\langle e, e - v^k \rangle + \rho \sum_{i \in \bar{I}} \langle |x_i^k|, v_i^k \rangle \right] \\ &\leq \lim_{k \rightarrow \infty} \left[\langle e, e - v^k \rangle + \rho \sum_{i \in I} \langle |x_i^k|, v_i^k \rangle + \rho \sum_{i \in \bar{I}} \langle |x_i^k|, v_i^k \rangle \right] \leq \alpha^*, \end{aligned}$$

where the first equality is using $\bar{v}_I = 0$, and the last equality is due to (7). This shows that (\bar{x}, \bar{v}) is a global optimal solution of (6). Thus, we prove that for any given $\rho > 0$, the problem (6) has a nonempty set of globally optimal solutions. \square

To show that the solution of problem (4) is equivalent to that of a single penalty problem (6), i.e., to prove that there exists $\bar{\rho} > 0$ such that the set of global optimal solutions of (4) coincides with that of (6) with $\rho > \bar{\rho}$, we need the following lemma. Since this lemma can be easily proved by contradiction, we here do not present its proof.

Lemma 3.2 *Given $M \in \mathbb{R}^{m \times n}$ and $q \in \mathbb{R}^m$. If $r = \min \{\|z\|_0 : \|Mz - q\| \leq \delta\} > 0$, then there exists $\alpha > 0$ such that for all z with $\|Mz - q\| \leq \delta$, we have $|z|_r^\downarrow > \alpha$, where $|z|_r^\downarrow$ means the r th component of $|z|^\downarrow$ which is the vector of components of $|z| \in \mathbb{R}^n$ being arranged in the nonincreasing order $|z|_1^\downarrow \geq |z|_2^\downarrow \geq \dots \geq |z|_n^\downarrow$.*

Now we are in a position to establish that (6) is an exact penalty problem of (4).

Theorem 3.1 *There exists a constant $\bar{\rho} > 0$ such that \mathcal{S}^* coincides with \mathcal{S}_ρ^* for all $\rho > \bar{\rho}$.*

Proof. Let $r = \min\{\|x\|_0 : \|Ax - b\| \leq \delta\}$. We only need to consider the case where $r > 0$ (if $r = 0$, the conclusion clearly holds for all $\rho > 0$). By Lemma 3.2, there exists $\alpha > 0$ such that for all x satisfying $\|Ax - b\| \leq \delta$, it holds that $|x|_r^\downarrow > \alpha$. This in turn means that $|x|_r^\downarrow > \alpha$ for all $(x, v) \in \mathcal{S}$ and $(x, v) \in \mathcal{S}_\rho$ with any $\rho > 0$.

Let (\bar{x}, \bar{v}) be an arbitrary point in \mathcal{S}^* . We prove that $(\bar{x}, \bar{v}) \in \mathcal{S}_\rho^*$ for all $\rho > 1/\alpha$, and then $\bar{\rho} = 1/\alpha$ is the one that we need. Let ρ be an arbitrary constant with $\rho > 1/\alpha$. Since $(\bar{x}, \bar{v}) \in \mathcal{S}^*$, from Proposition 2.1(a) and the equivalence between (4) and (1), it follows that $\bar{v} = e - \text{sign}(|\bar{x}|)$ and $\|\bar{x}\|_0 = r$. Note that $|x|_r^\downarrow > \alpha > 1/\rho$ for any $(x, v) \in \mathcal{S}_\rho$ since $\rho > 1/\alpha$. Hence, for any $(x, v) \in \mathcal{S}_\rho$, the following inequalities hold:

$$\begin{aligned} \langle e, e - v \rangle + \rho \langle v, |x| \rangle &= \sum_{i=1}^n (1 - v_i + \rho v_i |x|_i) \geq \sum_{|x|_i > 1/\rho} (1 - v_i + \rho v_i |x|_i) \\ &\geq r = \langle e, e - \bar{v} \rangle + \rho \langle \bar{v}, |\bar{x}| \rangle \end{aligned}$$

where the first inequality is using $1 - v_i + \rho v_i |x|_i \geq 0$ for all $i \in \{1, 2, \dots, n\}$, and the second inequality is since $|x|_r^\downarrow > 1/\rho$ and $0 \leq v_i \leq 1$. Since (x, v) is an arbitrary point in \mathcal{S}_ρ and $(\bar{x}, \bar{v}) \in \mathcal{S}_\rho$, the last inequality implies that $(\bar{x}, \bar{v}) \in \mathcal{S}_\rho^*$.

Next we show that if $(\bar{x}, \bar{v}) \in \mathcal{S}_\rho^*$ for $\rho > 1/\alpha$, then $(\bar{x}, \bar{v}) \in \mathcal{S}^*$. For convenience, let

$$I_- := \{i \in \{1, \dots, n\} \mid |\bar{x}|_i \leq \rho^{-1}\} \quad \text{and} \quad I_+ := \{i \in \{1, \dots, n\} \mid |\bar{x}|_i > \rho^{-1}\}.$$

Note that $1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i \geq 0$ for all i , and $1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i \geq 1$ for all $i \in I_+$. Also, the latter together with $|\bar{x}|_r^\downarrow > \alpha > 1/\rho$ implies $|I_+| \geq r$. Then, for any $(x, v) \in \mathcal{S}$ we have

$$\begin{aligned} \langle e, e - v \rangle &= \langle e, e - v \rangle + \rho \langle v, |x| \rangle \geq \langle e, e - \bar{v} \rangle + \rho \langle \bar{v}, |\bar{x}| \rangle \\ &= \sum_{i=1}^n (1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i) \geq \sum_{i \in I_+} (1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i) \geq r, \end{aligned}$$

where the first inequality is using $\mathcal{S} \subseteq \mathcal{S}_\rho$. Let $\tilde{x} \in \mathbb{R}^n$ be such that $\|\tilde{x}\|_0 = r$ and $\|A\tilde{x} - b\| \leq \delta$. Such \tilde{x} exists by the definition of r . Then $(\tilde{x}, e - \text{sign}(|\tilde{x}|)) \in \mathcal{S} \subseteq \mathcal{S}_\rho$, and from the last inequality $r = \langle e, e - (e - \text{sign}(|\tilde{x}|)) \rangle \geq \langle e, e - \bar{v} \rangle + \rho \langle \bar{v}, |\bar{x}| \rangle \geq r$. Thus,

$$r = \langle e, e - \bar{v} \rangle + \rho \langle \bar{v}, |\bar{x}| \rangle = \sum_{i \in I_-} (1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i) + \sum_{i \in I_+} (1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i).$$

Since $1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i \geq 0$ for $i \in I_-$, $1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i \geq 1$ for $i \in I_+$ and $|I_+| \geq r$, from the last equation it is not difficult to deduce that

$$1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i = 0 \quad \text{for } i \in I_-, \quad 1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i = 1 \quad \text{for } i \in I_+, \quad \text{and } |I_+| = r. \quad (8)$$

Since each $1 - \bar{v}_i + \rho \bar{v}_i |\bar{x}|_i$ is nonnegative, the first equality in (8) implies that

$$\bar{v}_i = 1 \quad \text{and} \quad |\bar{x}|_i = 0 \quad \text{for } i \in I_-;$$

while the second equality in (8) implies that $\bar{v}_i = 0$ for $i \in I_+$. Combining the last equation with $|I_+| = r$, we readily obtain $\|\bar{x}\|_0 = r$ and $\bar{v} = e - \text{sign}(|\bar{x}|)$. This, together with $\|A\bar{x} - b\| \leq \delta$, shows that $(\bar{x}, \bar{v}) \in \mathcal{S}^*$. Thus, we complete the proof. \square

Theorem 3.1 shows that to solve the MPEC problem (4), it suffices to solve (6) with a suitable large $\rho > \bar{\rho}$. Since the threshold $\bar{\rho}$ is unknown in advance, we need to solve a finite number of penalty problems with increasing ρ . Although problem (6) for a given $\rho > 0$ has a nonconvex objective function, its separable structure leads to an explicit solution with respect to the variable v if the variable x is fixed. This motivates us to propose the following decomposition method for (4), and consequently for (1).

Algorithm 3.1 (*Exact penalty decomposition method for problem (4)*)

(S.0) *Given a tolerance $\epsilon > 0$ and a ratio $\sigma > 1$. Choose an initial penalty parameter $\rho_0 > 0$ and a starting point $v^0 = e$. Set $k := 0$.*

(S.1) *Solve the following weighted l_1 -norm minimization problem*

$$x^{k+1} \in \arg \min_{x \in \mathbb{R}^n} \left\{ \langle v^k, |x| \rangle : \|Ax - b\| \leq \delta \right\}. \quad (9)$$

(S.2) *If $|x_i^{k+1}| > 1/\rho_k$, then set $v_i^{k+1} = 0$; and otherwise set $v_i^{k+1} = 1$.*

(S.3) *If $\langle v^{k+1}, |x^{k+1}| \rangle \leq \epsilon$, then stop; and otherwise go to (S.4).*

(S.4) *Let $\rho_{k+1} := \sigma \rho_k$ and $k := k + 1$, and then go to Step (S.1).*

Remark 3.1 *Note that the vector v^{k+1} in (S.2) is an optimal solution of the problem*

$$\min_{0 \leq v \leq e} \left\{ \langle e, e - v \rangle + \rho_k \langle |x^{k+1}|, v \rangle \right\}. \quad (10)$$

So, Algorithm 3.1 is solving the nonconvex penalty problem (6) in an alternating way.

The following lemma shows that the weighted l_1 -norm subproblem (9) has a solution.

Lemma 3.3 *For each fixed k , the subproblem (9) has an optimal solution.*

Proof. Note that the feasible set of (9) is \mathcal{F} , which is nonempty by the given assumption in the introduction, and its objective function is bounded below in the feasible set. Let ν^* be the infimum of the objective function of (9) on the feasible set. Then there exists a feasible sequence $\{x^l\}$ such that $\langle v^k, |x^l| \rangle \rightarrow \nu^*$ as $l \rightarrow \infty$. Let $I := \{i \mid v_i^k = 1\}$. Then, noting that $\langle v^k, |x^l| \rangle = \sum_{i \in I} |x_i^l|$, we have that the sequence $\{x_I^l\}$ is bounded. Without loss of generality, we assume that $x_I^l \rightarrow \tilde{x}_I$. Let $y^l = Ax^l - b$. Noting that $\|y^l\| \leq \delta$, we may assume that $\{y^l\}$ converges to \tilde{y} . Since the set $A_{\bar{I}} \mathbb{R}^{|\bar{I}|}$ is closed and $A_{\bar{I}} x_{\bar{I}}^l = y^l - A_I x_I^l + b$ for each l , where $\bar{I} = \{1, 2, \dots, n\} \setminus I$, there exists $\tilde{x}_{\bar{I}} \in \mathbb{R}^{|\bar{I}|}$ such

that $A_{\bar{T}}\tilde{x}_{\bar{T}} = \tilde{y} - A_I\tilde{x}_I + b$, i.e., $A_{\bar{T}}\tilde{x}_{\bar{T}} + A_I\tilde{x}_I - b = \tilde{y}$. Let $\tilde{x} = (\tilde{x}_I; \tilde{x}_{\bar{T}})$. Then, \tilde{x} is a feasible solution to (9) with $\langle v^k, \tilde{x} \rangle = \nu^*$. So, \tilde{x} is an optimal solution of (9). \square

For Algorithm 3.1, we can establish the following finite termination result.

Theorem 3.2 *Algorithm 3.1 will terminate after at most $\lceil \frac{\ln(n) - \ln(\epsilon\rho_0)}{\ln \sigma} \rceil$ iterations.*

Proof. By Lemma 3.3, for each $k \geq 0$ the subproblem (9) has a solution x^{k+1} . From Step (S.2) of Algorithm 3.1, we know that $v_i^{k+1} = 1$ for those i with $|x_i^{k+1}| \leq 1/\rho_k$. Then,

$$\langle v^{k+1}, |x^{k+1}| \rangle = \sum_{\{i: v_i^{k+1}=1\}} |x_i^{k+1}| \leq \frac{n}{\rho_k}.$$

This means that, when $\rho_k \geq \frac{n}{\epsilon}$, Algorithm 3.1 must terminate. Note that $\rho_k \geq \sigma^k \rho_0$. Therefore, Algorithm 3.1 will terminate when $\sigma^k \rho_0 \geq \frac{n}{\epsilon}$, i.e., $k \geq \lceil \frac{\ln(n) - \ln(\epsilon\rho_0)}{\ln \sigma} \rceil$. \square

We next focus on the theoretical results of Algorithm 3.1 for the case where $\delta = 0$. To this end, let x^* be an optimal solution of the zero-norm problem (1) and write

$$I^* = \{i \mid x_i^* \neq 0\} \quad \text{and} \quad \bar{I}^* = \{1, \dots, n\} \setminus I^*.$$

In addition, we also need the following null space condition for a given vector $v \in \mathbb{R}_+^n$:

$$\langle v_{I^*}, |y_{I^*}| \rangle < \langle v_{\bar{I}^*}, |y_{\bar{I}^*}| \rangle \quad \text{for any } 0 \neq y \in \text{Null}(A). \quad (11)$$

Theorem 3.3 *Assume that $\delta = 0$ and v^k satisfies the condition (11) for some nonnegative integer k . Then, $x^{k+1} = x^*$. If, in addition, v^{k+1} also satisfies the condition (11), then the vector v^{k+l} for all $l \geq 2$ satisfy the null space condition (11) and $x^{k+l+1} = x^*$ for all $l \geq 1$. Consequently, if v^0 satisfies the condition (11), then $x^k = x^*$ for all $k \geq 1$.*

Proof. We first prove the first part. Suppose that $x^{k+1} \neq x^*$. Let $y^{k+1} = x^{k+1} - x^*$. Clearly, $0 \neq y^{k+1} \in \text{Null}(A)$. Since v^k satisfies the condition (11), we have that

$$\langle v_{\bar{I}^*}^k, |y_{\bar{I}^*}^{k+1}| \rangle > \langle v_{I^*}^k, |y_{I^*}^{k+1}| \rangle. \quad (12)$$

On the other hand, from step (S.1), it follows that

$$\langle v^k, |x^*| \rangle \geq \langle v^k, |x^{k+1}| \rangle = \langle v^k, |x^* + y^{k+1}| \rangle \geq \langle v^k, |x^*| \rangle - \langle v_{I^*}^k, |y_{I^*}^{k+1}| \rangle + \langle v_{\bar{I}^*}^k, |y_{\bar{I}^*}^{k+1}| \rangle.$$

This implies that $\langle v_{I^*}^k, |y_{I^*}^{k+1}| \rangle \geq \langle v_{\bar{I}^*}^k, |y_{\bar{I}^*}^{k+1}| \rangle$. Thus, we obtain a contradiction to (12). Consequently, $x^{k+1} = x^*$. Since v^{k+1} also satisfies the null space condition (11), using the same arguments yields that $x^{k+2} = x^*$. We next show by induction that v^{k+l} for all $l \geq 2$ satisfy the condition (11) and $x^{k+l+1} = x^*$ for all $l \geq 2$. To this end, we define

$$I_\nu := \left\{ i \mid x_i^* > \frac{1}{\rho_{k+\nu}} \right\} \quad \text{and} \quad \bar{I}_\nu := \left\{ i \mid 0 < x_i^* \leq \frac{1}{\rho_{k+\nu}} \right\}$$

for any given nonnegative integer ν . Clearly, $I^* = I_\nu \cup \bar{I}_\nu$. Also, by noting that $\rho_{k+\nu+1} = \sigma \rho_{k+\nu}$ by (S.4) and $\sigma > 1$, we have that $I_\nu \subseteq I_{\nu+1} \subseteq I^*$. We first show that the result holds for $l = 2$. Since $x^{k+1} = x^*$, we have $v_{I_0}^{k+1} = 0$ and $v_{\bar{I}_0}^{k+1} = e$ by step (S.2). Since $x^{k+2} = x^*$, from step (S.2) it follows that $v_{I_1}^{k+2} = 0$ and $v_{\bar{I}_1}^{k+2} = e$. Now we obtain that

$$\begin{aligned} \langle v_{I^*}^{k+2}, |y_{I^*}| \rangle &= \langle v_{I_1}^{k+2}, |y_{I_1}| \rangle + \langle v_{\bar{I}_1}^{k+2}, |y_{\bar{I}_1}| \rangle = \langle v_{\bar{I}_1}^{k+2}, |y_{\bar{I}_1}| \rangle \leq \langle v_{\bar{I}_0}^{k+1}, |y_{\bar{I}_0}| \rangle \\ &= \langle v_{I_0}^{k+1}, |y_{I_0}| \rangle + \langle v_{\bar{I}_0}^{k+1}, |y_{\bar{I}_0}| \rangle = \langle v_{I^*}^{k+1}, |y_{I^*}| \rangle \\ &< \langle v_{\bar{I}^*}^{k+1}, |y_{\bar{I}^*}| \rangle = \langle v_{\bar{I}^*}^{k+2}, |y_{\bar{I}^*}| \rangle \end{aligned} \quad (13)$$

for any $0 \neq y \in \text{Null}(A)$, where the first equality is due to $I^* = I_1 \cup \bar{I}_1$, the second equality is using $v_{I_1}^{k+2} = 0$, the first inequality is due to $\bar{I}_1 \subseteq \bar{I}_0$, $v_{\bar{I}_1}^{k+2} = e$ and $v_{\bar{I}_0}^{k+1} = e$, the second inequality is using the assumption that $v_{\bar{I}^*}^{k+1}$ satisfies the null space condition (11), and the last equality is due to $v_{I^*}^{k+1} = e$ and $v_{\bar{I}^*}^{k+2} = e$. The inequality (13) shows that $v_{\bar{I}^*}^{k+2}$ satisfies the null space condition (11), and using the same arguments as for the first part yields that $x^{k+3} = x^*$. Now assuming that the result holds for $l(\geq 2)$, we show that it holds for $l+1$. Indeed, using the same arguments as above, we obtain that

$$\begin{aligned} \langle v_{I^*}^{k+l+1}, |y_{I^*}| \rangle &= \langle v_{I_l}^{k+l+1}, |y_{I_l}| \rangle + \langle v_{\bar{I}_l}^{k+l+1}, |y_{\bar{I}_l}| \rangle = \langle v_{\bar{I}_l}^{k+l+1}, |y_{\bar{I}_l}| \rangle \leq \langle v_{\bar{I}_{l-1}}^{k+l}, |y_{\bar{I}_{l-1}}| \rangle \\ &= \langle v_{I_{l-1}}^{k+l}, |y_{I_{l-1}}| \rangle + \langle v_{\bar{I}_{l-1}}^{k+l}, |y_{\bar{I}_{l-1}}| \rangle = \langle v_{I^*}^{k+l}, |y_{I^*}| \rangle \\ &< \langle v_{\bar{I}^*}^{k+l}, |y_{\bar{I}^*}| \rangle = \langle v_{\bar{I}^*}^{k+l+1}, |y_{\bar{I}^*}| \rangle \end{aligned} \quad (14)$$

for any $0 \neq y \in \text{Null}(A)$. This shows that $v_{\bar{I}^*}^{k+l+1}$ satisfies the null space condition (11), and using the same arguments as the first part yields that $x^{k+l+2} = x^*$. Thus, we show that $v_{\bar{I}^*}^{k+l}$ for all $l \geq 2$ satisfy the condition (11) and $x^{k+l+1} = x^*$ for all $l \geq 2$. \square

Theorem 3.3 shows that, when $\delta = 0$, if there are two successive vectors v^k and v^{k+1} satisfy the null space condition (11), then the iterates after x^k are all equal to some optimal solution of (1). Together with Theorem 3.2, this means that Algorithm 3.1 can find an optimal solution of (1) within a finite number of iterations under (11). To the best of our knowledge, the condition (11) is first proposed by Khajehnejad et al. [23], which generalizes the null space condition of [34] to the case of weighted l_1 -norm minimization and is weaker than the truncated null space condition [38, Definition 1].

4 Solution of weighted l_1 -norm subproblems

This section is devoted to the solution of the subproblems involved in Algorithm 3.1:

$$\min_{x \in \mathbb{R}^n} \{ \langle v, |x| \rangle : \|Ax - b\| \leq \delta \} \quad (15)$$

where $v \in \mathbb{R}^n$ is a given nonnegative vector. For this problem, one may reformulate it as a linear programming problem (for $\delta = 0$) or a second-order cone programming

problem (for $\delta > 0$), and then directly apply the interior point method software SeDuMi [35] or l_1 -MAGIC [9] for solving it. However, such second-order type methods are time-consuming, and are not suitable for handling large-scale problems. Motivated by the recent work [22], we in this section develop a partial proximal point algorithm (PPA) for the following reformulation of the weighted l_1 -norm problem (15):

$$\min_{u \in \mathbb{R}^m, x \in \mathbb{R}^n} \left\{ \langle v, |x| \rangle + \frac{\beta}{2} \|u\|^2 : Ax + u = b \right\} \quad \text{for some } \beta > 0. \quad (16)$$

Clearly, (16) is equivalent to (15) if $\delta > 0$; and otherwise is a penalty problem of (15).

Given a starting point $(u^0, x^0) \in \mathbb{R}^m \times \mathbb{R}^n$, the partial PPA for (16) consists of solving approximately a sequence of strongly convex minimization problems

$$(u^{k+1}, x^{k+1}) \approx \arg \min_{u \in \mathbb{R}^m, x \in \mathbb{R}^n} \left\{ \langle v, |x| \rangle + \frac{\beta}{2} \|u\|^2 + \frac{1}{2\lambda_k} \|x - x^k\|^2 : Ax + u = b \right\}, \quad (17)$$

where $\{\lambda_k\}$ is a sequence of parameters satisfying $0 < \lambda_k \uparrow \bar{\lambda} \leq +\infty$. For the global and local convergence of this method, the interested readers may refer to Ha's work [20], where he first considered such PPA for finding a solution of generalized equations. Here we focus on the approximate solution of the subproblems (17) via the dual method.

Let $L : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ denote the Lagrangian function of the problem (17)

$$L(u, x, y) := \langle v, |x| \rangle + \frac{\beta}{2} \|u\|^2 + \frac{1}{2\lambda_k} \|x - x^k\|^2 + \langle y, Ax + u - b \rangle.$$

Then the minimization problem (17) is expressed as $\min_{(u,x) \in \mathbb{R}^m \times \mathbb{R}^n} \sup_{y \in \mathbb{R}^m} L(u, x, y)$. Also, by [33, Corollary 37.3.2] and the coercivity of L with respect to u and x ,

$$\min_{(u,x) \in \mathbb{R}^m \times \mathbb{R}^n} \sup_{y \in \mathbb{R}^m} L(u, x, y) = \sup_{y \in \mathbb{R}^m} \min_{(u,x) \in \mathbb{R}^m \times \mathbb{R}^n} L(u, x, y). \quad (18)$$

This means that there is no dual gap between the problem (17) and its dual problem

$$\sup_{y \in \mathbb{R}^m} \min_{(u,x) \in \mathbb{R}^m \times \mathbb{R}^n} L(u, x, y). \quad (19)$$

Hence, we can obtain the approximate optimal solution (u^{k+1}, x^{k+1}) of (17) by solving (19). To give the expression of the objective function of (19), we need the following operator $S_\lambda(\cdot, v) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ associated to the vector v and any $\lambda > 0$:

$$S_\lambda(z, v) := \arg \min_{x \in \mathbb{R}^n} \left\{ \langle v, |x| \rangle + \frac{1}{2\lambda} \|x - z\|^2 \right\}.$$

An elementary computation yields the explicit expression of the operator $S_\lambda(\cdot, v)$:

$$S_\lambda(z, v) = \text{sign}(z) \odot \max\{|z| - \lambda v, 0\} \quad \forall z \in \mathbb{R}^n, \quad (20)$$

where “ \odot ” means the componentwise product of two vectors, and for any $z \in \mathbb{R}^n$,

$$\begin{aligned}\langle v, |S_\lambda(z, v)| \rangle &= \langle \text{sign}(z) \odot v, S_\lambda(z, v) \rangle, \\ \langle S_\lambda(z, v), S_\lambda(z, v) \rangle &= \langle S_\lambda(z, v), z \rangle - \lambda \langle \text{sign}(z) \odot v, S_\lambda(z, v) \rangle.\end{aligned}\quad (21)$$

From the definition of $S_\lambda(\cdot, v)$ and equation (21), we immediately obtain that

$$\min_{(u, x) \in \mathbb{R}^m \times \mathbb{R}^n} L(u, x, y) = -b^T y - \frac{1}{2\beta} \|y\|^2 - \frac{1}{2\lambda_k} \|S_{\lambda_k}(x^k - \lambda_k A^T y, v)\|^2 + \frac{1}{2\lambda_k} \|x^k\|^2.$$

Consequently, the dual problem (19) is equivalent to the following minimization problem

$$\min_{y \in \mathbb{R}^m} \Phi(y) := b^T y + \frac{1}{2\beta} \|y\|^2 + \frac{1}{2\lambda_k} \|S_{\lambda_k}(x^k - \lambda_k A^T y, v)\|^2. \quad (22)$$

The following lemma summarizes the favorable properties of the function Φ .

Lemma 4.1 *The function Φ defined by (22) has the following properties:*

(a) Φ is a continuously differentiable convex function with gradient given by

$$\nabla \Phi(y) = b + \beta^{-1} y - A S_{\lambda_k}(x^k - \lambda_k A^T y, v) \quad \forall y \in \mathbb{R}^m.$$

(b) If \hat{y}^k is a root to the system $\nabla \Phi(y) = 0$, then $(\hat{u}^{k+1}, \hat{x}^{k+1})$ defined by

$$\hat{u}^{k+1} := -\beta^{-1} \hat{y}^k \quad \text{and} \quad \hat{x}^{k+1} := S_{\lambda_k}(x^k - \lambda_k A^T \hat{y}^k, v)$$

is the unique optimal solution of the primal problem (17).

(c) The gradient mapping $\nabla \Phi(\cdot)$ is Lipschitz continuous and strongly semismooth.

(d) The Clarke’s generalized Jacobian of the mapping $\nabla \Phi$ at any point y satisfies

$$\partial(\nabla \Phi)(y) \subseteq \beta^{-1} I + \lambda_k A \partial_x H(z, v) A^T := \widehat{\partial}^2 \Phi(y). \quad (23)$$

where $z = x^k - \lambda_k A^T y$ and $H(x, v) := \text{sign}(x) \odot \max\{|x| - \lambda_k v, 0\}$.

Proof. (a) By the definition of $S_\lambda(\cdot, v)$ and equation (21), it is not hard to verify that

$$\frac{1}{2\lambda} \|S_\lambda(z, v)\|^2 = \frac{1}{2\lambda} \|z\|^2 - \min_{x \in \mathbb{R}^n} \left\{ \langle v, |x| \rangle + \frac{1}{2\lambda} \|x - z\|^2 \right\}.$$

Note that the second term on the right hand side is the Moreau-Yosida regularization of the convex function $f(x) := \langle v, |x| \rangle$. From [33] it follows that $\|S_\lambda(\cdot, v)\|^2$ is continuously differentiable, which implies that Φ is continuously differentiable.

(b) Note that \hat{y}^k is an optimal solution of (19) and there is no dual gap between the primal problem (17) and its dual (19) by equation (18). The desired result then follows.

(c) The result is immediate by the expression of Φ and $S_{\lambda_k}(\cdot, v)$.

(d) The result is implied by the corollary in [7, p.75]. Notice that the inclusion in (23) can not be replaced by the equality since A is assumed to be of full row rank. \square

Remark 4.1 For a given $v \in \mathbb{R}_+^n$, from [7, Chaper 2] we know that the Clarke Jacobian of the mapping $H(\cdot, v)$ defined in Lemma 4.1(d) takes the following form

$$\partial_z H(z, v) = \partial\phi(z_1) \times \partial\phi(z_2) \times \cdots \times \partial\phi(z_n)$$

$$\text{with } \partial\phi(z_i) = \{1\} \text{ if } v_i = 0 \text{ and otherwise } \partial\phi(z_i) = \begin{cases} \{1\} & \text{if } |z_i| > \lambda v_i, \\ [0, 1] & \text{if } |z_i| = \lambda v_i, \\ \{0\} & \text{if } |z_i| < \lambda v_i. \end{cases}$$

By Lemma 4.1(a) and (b), we can apply the limited-memory BFGS algorithm [30] for solving (22), but the direction yielded by this method may not approximate the Newton direction well if the elements in $\widehat{\partial}^2\Phi(y^k)$ are badly scaled since Φ is only once continuously differentiable. So, we need some Newton steps to bring in the second-order information. In view of Lemma 4.1(c) and (d), we apply the semismooth Newton method [31] for finding a root of the nonsmooth system $\nabla\Phi(y) = 0$. To make it possible to solve large-scale problems, we use the conjugate gradient (CG) method to yield approximate Newton steps. This leads to the following semismooth Newton-CG method.

Algorithm 4.1 (The semismooth Newton-CG method for (19))

(S0) Given $\bar{\epsilon} > 0$, $j_{\max} > 0$, $\tau_1, \tau_2 \in (0, 1)$, $\varrho \in (0, 1)$ and $\mu \in (0, \frac{1}{2})$. Choose a starting point $y^0 \in \mathbb{R}^m$ and set $j := 0$.

(S1) If $\|\nabla\Phi(y^j)\| \leq \bar{\epsilon}$ or $j > j_{\max}$, then stop. Otherwise, go to the next step.

(S2) Apply the CG method to seek an approximate solution d^j to the linear system

$$(V^j + \varepsilon^j I)d = -\nabla\Phi(y^j), \quad (24)$$

where $V^j \in \widehat{\partial}^2\Phi(y^j)$ with $\widehat{\partial}^2\Phi(\cdot)$ given by (23), and $\varepsilon^j := \tau_1 \min\{\tau_2, \|\nabla\Phi(y^j)\|\}$.

(S3) Seek the smallest nonnegative integer l_j such that the following inequality holds:

$$\Phi(y^j + \varrho^{l_j} d^j) \leq \Phi(y^j) + \mu \varrho^{l_j} \langle \nabla\Phi(y^j), d^j \rangle.$$

(S4) Set $y^{j+1} := y^j + \varrho^{l_j} d^j$ and $j := j + 1$, and then go to Step (S.1).

From the definition of $\widehat{\partial}^2\Phi(\cdot)$ in Lemma 4.1(d), V^j in Step (S2) of Algorithm 4.1 is positive definite, and consequently the search direction d^j is always a descent direction. For the global convergence and the rate of local convergence of Algorithm 4.1, the interested readers may refer to [40]. Once we have an approximate optimal y^k of (19), the approximate optimal solution (u^{k+1}, x^{k+1}) of (17) is obtained from the formulas

$$u^{k+1} := -\beta^{-1}y^k \quad \text{and} \quad x^{k+1} := S_{\lambda_k}(x^k - \lambda_k A^T y^k, v).$$

To close this section, we take a look at the selection of V^j in Step (S2) for numerical experiments of the next section. By the definition of $\widehat{\partial}^2\Phi(\cdot)$, V^j takes the form of

$$V^j = \beta^{-1}I + \lambda_k A D^k A^T, \quad (25)$$

where $D^k \in \partial_x H(z, v)$ with $z = x^k - \lambda_k A^T y$. By Remark 4.1, D^k is a diagonal matrix, and we select the i th diagonal element $D_i^k = 1$ if $|z_i| \geq \lambda v_i$ and otherwise $D_i^k = 0$.

5 Numerical experiments

In this section, we test the performance of Algorithm 3.1 with the subproblem (9) solved by the partial PPA. Notice that using the L-BFGS or the semismooth Newton-CG alone to solve the subproblem (17) of the partial PPA can not yield the desired result, since using the L-BFGS alone will not yield good feasibility for those difficult problems due to the lack of the second-order information of objective function, while using the semismooth Newton-CG alone will meet difficulty for the weighted l_1 -norm subproblems involved in the beginning of Algorithm 3.1. In view of this, we develop an exact penalty decomposition algorithm with the subproblem (9) solved by the partial PPA, for which the subproblems (17) are solved by combining the L-BFGS with the semismooth Newton-CG. The detailed iteration steps of the whole algorithm are described as follows, where for any given $\beta_k, \lambda_k > 0$ and $(x^k, v^k) \in \mathbb{R}^n \times \mathbb{R}^n$, the function $\Phi_k : \mathbb{R}^m \rightarrow \mathbb{R}$ is defined as

$$\Phi_k(y) := b^T y + \frac{1}{2\beta_k} \|y\|^2 + \frac{1}{2\lambda_k} \|S_{\lambda_k}(x^{k-1} - \lambda_k A^T y, v^{k-1})\|^2 \quad \forall y \in \mathbb{R}^m.$$

Algorithm 5.1 (Practical exact penalty decomposition method for (1))

(S.0) *Given $\epsilon, \epsilon_1 > 0$, $\omega_1, \omega_2 > 0$, $\gamma \in (0, 1)$, $\sigma \geq 1$ and $\underline{\lambda} > 0$. Choose a sufficiently large β_0 and suitable $\lambda_0 > 0$ and $\rho_0 > 0$. Set $(x^0, v^0, y^0) = (0, e, e)$ and $k = 0$.*

(S.1) **While** $\frac{\|Ax^k - b\|}{\max\{1, \|b\|\}} > \epsilon_1$ *and* $\lambda_k > \underline{\lambda}$ **do**

- Set $\lambda_{k+1} = \gamma^k \lambda_0$ and $\beta_{k+1} = \beta_k$.
- With y^k as the starting point, find $y^{k+1} \approx \arg \min_{y \in \mathbb{R}^m} \Phi_{k+1}(y)$ such that $\|\nabla \Phi_{k+1}(y^{k+1})\| \leq \omega_1$ by using the L-BFGS algorithm.
- Set $x^{k+1} := S_{\lambda_{k+1}}(x^k - \lambda_{k+1} A^T y^{k+1}, v^k)$ and $v_i^{k+1} := \begin{cases} 0 & \text{if } x_i^{k+1} > \rho_k^{-1}, \\ 1 & \text{otherwise.} \end{cases}$
- Set $\rho_{k+1} = \sigma \rho_k$ and $k := k + 1$.

End

(S.2) **While** $\frac{\|Ax^k - b\|}{\max\{1, \|b\|\}} > \epsilon_1$ *or* $\langle v^k, |x^k| \rangle > \epsilon$ **do**

- Set $\lambda_{k+1} = \lambda_k$ and $\beta_{k+1} = \beta_k$.
- With y^k as the starting point, find $y^{k+1} \approx \arg \min_{y \in \mathbb{R}^m} \Phi_{k+1}(y)$ such that $\|\nabla \Phi_{k+1}(y^{k+1})\| \leq \omega_2$ by using Algorithm 4.1.
- Set $x^{k+1} := S_{\lambda_{k+1}}(x^k - \lambda_{k+1} A^T y^{k+1}, v^k)$ and $v_i^{k+1} := \begin{cases} 0 & \text{if } x_i^{k+1} > \rho_k^{-1}, \\ 1 & \text{otherwise.} \end{cases}$
- Set $\rho_{k+1} = \sigma \rho_k$ and $k := k + 1$.

End

By the choice of starting point (x^0, v^0, y^0) , the first step of Algorithm 5.1 is solving

$$\min_{u \in \mathbb{R}^m, x \in \mathbb{R}^n} \left\{ \|x\|_1 + \frac{\beta_0}{2} \|u\|^2 + \frac{1}{2\lambda_0} \|x\|^2 : Ax + u = b \right\},$$

whose solution is the minimum-norm solution of the l_1 -norm minimization problem

$$\min_{x \in \mathbb{R}^n} \left\{ \|x\|_1 : Ax = b \right\} \quad (26)$$

if β_0 and λ_0 are chosen to be sufficiently large (see [27]). Taking into account that the l_1 -norm minimization problem is a good convex surrogate for the zero-norm minimization problem (1), we should solve the problem $\min_{y \in \mathbb{R}^m} \Phi_1(y)$ as well as we can. If the initial step can not yield an iterate with good feasibility, then we solve the regularized problems

$$\min_{x \in \mathbb{R}^n, u \in \mathbb{R}^m} \left\{ \langle v^k, |x| \rangle + \frac{\beta_{k+1}}{2} \|u\|^2 + \frac{1}{2\lambda_{k+1}} \|x - x^k\|^2 : Ax + u = b \right\}. \quad (27)$$

with a decreasing sequence $\{\lambda_k\}$ and a nondecreasing sequence $\{\beta_k\}$ via the L-BFGS. Once a good feasible point is found in Step (S.1), Algorithm 5.1 turns to the second stage, i.e., to solve (27) with nondecreasing sequences $\{\beta_k\}$ and $\{\lambda_k\}$ via Algorithm 4.1.

Unless otherwise stated, the parameters involved in Algorithm 5.1 were chosen as:

$$\begin{aligned} \epsilon &= \frac{10^{-2}}{\max(1, \|b\|)}, \epsilon_1 = 10^{-6}, \omega_1 = 10^{-5}, \omega_2 = 10^{-6}, \underline{\lambda} = 10^{-2}, \sigma = 2, \\ \beta_0 &= \max(5\|b\| \times 10^6, 10^{10}), \rho_0 = \min(1, 10/\|b\|), \lambda_0 = \hat{\gamma}\|b\|, \end{aligned} \quad (28)$$

where we set $\gamma = 0.6$ and $\hat{\gamma} = 5$ if A is stored implicitly (i.e., A is given in operator form); and otherwise we chose γ and $\hat{\gamma}$ by the scale of the problem, i.e.,

$$\gamma = \begin{cases} 0.5 & \text{if } \|b\| > 10^5 \text{ or } \|b\| \leq 5, \\ 0.8 & \text{otherwise,} \end{cases} \quad \text{and} \quad \hat{\gamma} = \begin{cases} 10 & \text{if } \|b\| > 10^5 \text{ or } \|b\| \leq 5, \\ 1.5 & \text{otherwise.} \end{cases}$$

We employed the L-BFGS with 5 limited-memory vector-updates and the nonmonotone Armijo line search rule [19] to yield an approximate solution to the minimization problem in Step (S.1) of Algorithm 5.1. Among others, the number of maximum iterations of the L-BFGS was chosen as **300** for the minimization of $\Phi_1(y)$, and **50** for the minimization of $\Phi_k(y)$ with $k \geq 2$. The parameters involved in Algorithm 4.1 are set as:

$$\bar{\epsilon} = 10^{-6}, j_{\max} = 50, \tau_1 = 0.1, \tau_2 = 10^{-4}, \varrho = 0.5, \mu = 10^{-4}. \quad (29)$$

In addition, during the testing, if the decrease of gradient is slow in Step (S.1), we terminate the L-BFGS in advance and then turn to the solution of the next subproblem. Unless otherwise stated, the parameters in **QPDM** and **ISDM** are all set to default values, the “Hybridls” type line search and “lbfgs” type subspace optimization method are chosen for **FPC_AS**, and $\mu = 10^{-10}$, $\epsilon = 10^{-12}$ and $\epsilon_x = 10^{-16}$ are used for **FPC_AS**.

All tests described in this section were run in MATLAB R2012(a) under a Windows operating system on an Intel Core(TM) i3-2120 3.30GHz CPU with 3GB memory.

To verify the effectiveness of Algorithm 5.1, we compared it with **QPDM** [25], **ISDM** [38] and **FPC_AS** on four different sets of problems. Since the four solvers return solutions with tiny but nonzero entries that can be regarded as zero, we use **nnzx** to denote the number of nonzeros in x which we estimate as in [2] by the minimum cardinality of a subset of the components of x that account for 99.9% of $\|x\|_1$; i.e.,

$$\mathbf{nnzx} := \min \left\{ \kappa : \sum_{i=1}^{\kappa} |x|_i^{\downarrow} \geq 0.999 \|x\|_1 \right\}.$$

Suppose that the exact sparsest solution x^* is known. We also compare the support of x^f with that of x^* , where x^f is the final iterate yielded by the above four solvers. To this end, we first remove tiny entries of x^f by setting all of its entries with a magnitude smaller than $0.1|x^*|_{\text{snz}}$ to zero, where $|x^*|_{\text{snz}}$ is the smallest nonzero component of $|x^*|$, and then compute the quantities “**sgn**”, “**miss**” and “**over**”, where

$$\mathbf{sgn} := |\{i \mid x_i^f x_i^* < 0\}|, \quad \mathbf{miss} := |\{i \mid x_i^f = 0, x_i^* \neq 0\}|, \quad \mathbf{over} := |\{i \mid x_i^f \neq 0, x_i^* = 0\}|.$$

5.1 Recoverability for some “pathological” problems

We tested Algorithm 5.1, **FPC_AS**, **ISDM** and **QPDM** on a set of small-scale, pathological problems described in Table 1. The first test set includes four problems Caltech Test 1, ..., Caltech Test 4 given by Candès and Becker, which, as mentioned in [39], are pathological because the magnitudes of the nonzero entries of the exact solution x^* lies in a large range. Such pathological problems are exaggerations of a large number of realistic problems in which the signals have both large and small entries. The second test set includes six problems Ameth6Xmeth20-Ameth6Xmeth24 and Ameth6Xmeth6 from [39], which are difficult since the number of nonzero entries in their solutions is close to the limit where the zero-norm problem (1) is equivalent to the l_1 -norm problem.

The numerical results of four solvers are reported in Table 2, where **nMat** means the total number of matrix-vector products involving A and A^T , **Time** means the computing time in seconds, **Res** denotes the l_2 -norm of recovered residual, i.e., $\mathbf{Res} = \|Ax^f - b\|$, and **Relerr** means the relative error between the recovered solution x^f and the true solution x^* , i.e., $\mathbf{Relerr} = \|x^f - x^*\|/\|x^*\|$. Since **ISDM** and **QPDM** do not record the number of matrix-vector products involving A and A^T , we mark **nMat** as “_”.

Table 2 shows that among the four solvers, **QPDM** has the worst performance and can not recovery any one of these problems, **ISDM** requires the most computing time and yields solutions with incorrect **miss** for the first test set, and Algorithm 5.1 and **FPC_AS** have comparable performance in terms of recoverability and computing time.

Table 1: Description of some pathological problems

ID	Name	n	m	K	(Magnitude, num. of entries on this level)
1	CaltechTest1	512	128	38	$(10^5, 33), (1, 5)$
2	CaltechTest2	512	128	37	$(10^5, 32), (1, 5)$
3	CaltechTest3	512	128	32	$(10^5, 31), (10^{-6}, 1)$
4	CaltechTest4	512	102	26	$(10^4, 13), (1, 12), (10^{-2}, 1)$
5	Ameth6Xmeth20	1024	512	150	$(1, 150)$
6	Ameth6Xmeth21	1024	512	151	$(1, 150)$
7	Ameth6Xmeth22	1024	512	152	$(1, 150)$
8	Ameth6Xmeth23	1024	512	153	$(1, 150)$
9	Ameth6Xmeth24	1024	512	154	$(1, 150)$
10	Ameth6Xmeth6	1024	512	154	$(1, 150)$

5.2 Sparse signal recovery from noiseless measurements

In this subsection we compare the performance of Algorithm 5.1 with that of **FPC_AS**, **ISDM** and **QPDM** for compressed sensing reconstruction on randomly generated problems. Given the dimension n of a signal, the number of observations m and the number of nonzeros K , we generated a random matrix $A \in \mathbb{R}^{m \times n}$ and a random $x^* \in \mathbb{R}^n$ in the same way as in [39]. Specifically, we generated a matrix by one of the following types:

Type 1: Gaussian matrix whose elements are generated independently and identically distributed from the normal distribution $N(0, 1)$;

Type 2: Orthogonalized Gaussian matrix whose rows are orthogonalized using a QR decomposition;

Type 3: Bernoulli matrix whose elements are ± 1 independently with equal probability;

Type 4: Hadamard matrix H , which is a matrix of ± 1 whose columns are orthogonal;

Type 5: Discrete cosine transform (DCT) matrix;

and then randomly selected m rows from this matrix to construct the matrix A . Similar to [39], we also scaled the matrix A constructed from matrices of types 1, 3, and 4 by the largest eigenvalue of AA^T . In order to generate the signal x^* , we first generated the support by randomly selecting K indexed between 1 and n , and then assigned a value to x_i^* for each i in the support by one of the following six methods:

Type 1: A normally distributed random variable (Gaussian signal);

Type 2: A uniformly distributed random variable in $(-1, 1)$;

Type 3: One (zero-one signal);

Table 2: Numerical results of four solvers for the pathological problems

ID	Solver	time(s)	Relerr	Res	nMat	nnzx	(sgn, miss, over)
1	Algorithm 5.1	0.39	5.16e-12	8.87e-9	1057	33	(0, 0, 0)
	FPC_AS	0.47	4.98e-12	4.37e-8	437	33	(0, 0, 0)
	ISDM	0.47	4.52e-6	9.97e-1	–	33	(0, 5, 0)
	QPDM	0.15	2.07e-0	1.53e-9	–	125	(0, 28, 118)
2	Algorithm 5.1	0.11	8.25e-14	8.44e-9	1060	32	(0, 0, 0)
	FPC_AS	0.12	1.86e-13	5.58e-8	357	32	(0, 0, 0)
	ISDM	0.37	4.27e-6	9.34e-1	–	32	(0, 5, 0)
	QPDM	0.01	2.18e-0	1.17e-9	–	123	(0, 28, 119)
3	Algorithm 5.1	0.16	4.56e-9	2.09e-14	1199	31	(0, 0, 0)
	FPC_AS	0.06	1.15e-9	1.61e-9	247	31	(0, 0, 0)
	ISDM	0.42	9.78e-7	4.67e-7	–	31	(0, 1, 0)
	QPDM	0.03	9.78e-7	4.67e-7	–	31	(0, 1, 0)
4	Algorithm 5.1	0.16	3.10e-7	4.05e-3	985	13	(0, 1, 0)
	FPC_AS	0.17	4.52e-13	7.51e-9	572	13	(0, 0, 0)
	ISDM	0.34	9.96e-5	1.38e-0	–	13	(0, 12, 1)
	QPDM	0.02	2.10e-0	6.00e-11	–	13	(0, 21, 97)
5	Algorithm 5.1	0.47	4.93e-14	5.69e-13	1098	150	(0, 0, 0)
	FPC_AS	0.25	6.80e-10	4.01e-9	412	150	(0, 0, 0)
	ISDM	3.17	6.67e-1	3.72e-1	–	464	(0, 15, 185)
	QPDM	1.58	8.65e-1	4.38e-1	–	492	(0, 28, 287)
6	Algorithm 5.1	0.36	4.91e-14	5.68e-13	730	151	(0, 0, 0)
	FPC_AS	0.29	6.96e-10	4.11e-9	408	151	(0, 0, 0)
	ISDM	3.38	4.92e-14	5.80e-14	–	151	(0, 0, 0)
	QPDM	0.92	6.58e-1	4.92e-1	–	480	(0, 16, 211)
7	Algorithm 5.1	0.41	4.91e-14	5.69e-13	910	152	(0, 0, 0)
	FPC_AS	0.36	8.10e-10	4.81e-9	461	152	(0, 0, 0)
	ISDM	3.29	5.02e-14	5.80e-13	–	152	(0, 0, 0)
	QPDM	1.42	6.89e-1	5.00e-1	–	481	(0, 21, 222)
8	Algorithm 5.1	0.34	4.95e-14	5.68e-13	809	153	(0, 0, 0)
	FPC_AS	0.34	9.19e-10	5.44e-9	578	153	(0, 0, 0)
	ISDM	2.96	5.51e-14	5.88e-13	–	153	(0, 0, 0)
	QPDM	1.36	2.06e-0	2.50e-1	–	494	(0, 37, 361)
9	Algorithm 5.1	0.37	4.94e-14	5.67e-13	826	154	(0, 0, 0)
	FPC_AS	0.41	9.41e-10	5.57e-9	572	154	(0, 0, 0)
	ISDM	3.81	4.70e-14	5.79e-13	–	154	(0, 0, 0)
	QPDM	1.47	2.71e-0	2.31e-1	–	496	(0, 42, 374)
10	Algorithm 5.1	0.61	4.94e-14	5.67e-8	1419	154	(0, 0, 0)
	FPC_AS	0.31	2.59e-13	1.57e-7	577	154	(0, 0, 0)
	ISDM	3.28	4.70e-14	5.79e-8	–	154	(0, 0, 0)
	QPDM	0.22	3.01e-0	2.22e+4	–	499	(0, 81, 420)

Type 4: The sign of a normally distributed random variable;

Type 5: A signal x with power-law decaying entries (known as compressible sparse signals) whose components satisfy $|x_i| \leq c_x i^{-p}$, where $c_x = 10^5$ and $p = 1.5$;

Type 6: A signal x with exponential decaying entries whose components satisfy

$$|x_i| \leq c_x e^{-pi} \quad \text{with } c_x = 1 \text{ and } p = 0.005.$$

Finally, the observation b was computed as $b = Ax^*$. The matrices of types 1, 2, 3 and 4 were stored explicitly, and the matrices of type 5 were stored implicitly. Unless otherwise stated, in the sequel, we call a signal recovered successfully by a solver if the relative error between the solution x^f generated and the original signal x^* is less than 5×10^{-7} .

We first took the matrix of type 1 for example to test the influence of the number of measurements m on the recoverability of four solvers for different types of signals. For each type of signal, we considered the dimension $n = 600$ and the number of nonzeros $K = 40$ and took the number of measurements $m \in \{80, 90, 100, \dots, 220\}$. For each m , we generated **50** problems randomly, and tested the frequency of successful recovery for each solver. The curves of Figure 1 depict how the recoverability of four solvers vary with the number of measurements for different types of signals.

Figure 1 shows that among the four solvers, **QPDM** has the worst recoverability for all six different types of signals, **ISDM** has a little better recoverability than Algorithm 5.1 for the signals of types 1, 2 and 6, which are much better than that of **FPC_AS**, and Algorithm 5.1, **FPC_AS** and **ISDM** have comparable recoverability for the signals of types 3 and 4. For the signals of type 5, Algorithm 5.1 has much better recoverability than **ISDM** and **FPC_AS**. After further testing, we found that for other types of A , the four solvers display the similar performance as in Figure 1 for the six kinds of signals (see Figure 2 for type 2), and **ISDM** requires the most computing time among the solvers.

Then we took the matrices of type 5 for example to show how the performance of Algorithm 5.1, **FPC_AS** and **ISDM** scales with the size of the problem. Since Figure 1-2 illustrates that the three solvers have the similar performance for the signals of types 1, 2 and 6, and the similar performance for the signals of type 3 and 4, we compared their performance only for the signals of types 1, 3 and 5. For each type of x^* , we generated **50** problems randomly for each $n \in \{2^7, 2^8, \dots, 2^{16}\}$ and tested the frequency and the average time of successful recovery for the three solvers, where $m = \mathbf{round}(n/6)$ for the signals of type 1, $m = \mathbf{round}(n/3)$ for the signals of type 3, and $m = \mathbf{round}(n/4)$ for the signals of type 5, and the number of nonzeros K was set to $\mathbf{round}(0.3m)$. The curves of Figure 3 depict how the recoverability and the average time of successful recovery vary with the size of the problem. When there is no signal recovered successfully, we replace the average time of successful recovery by the average computing time of 50 problems.

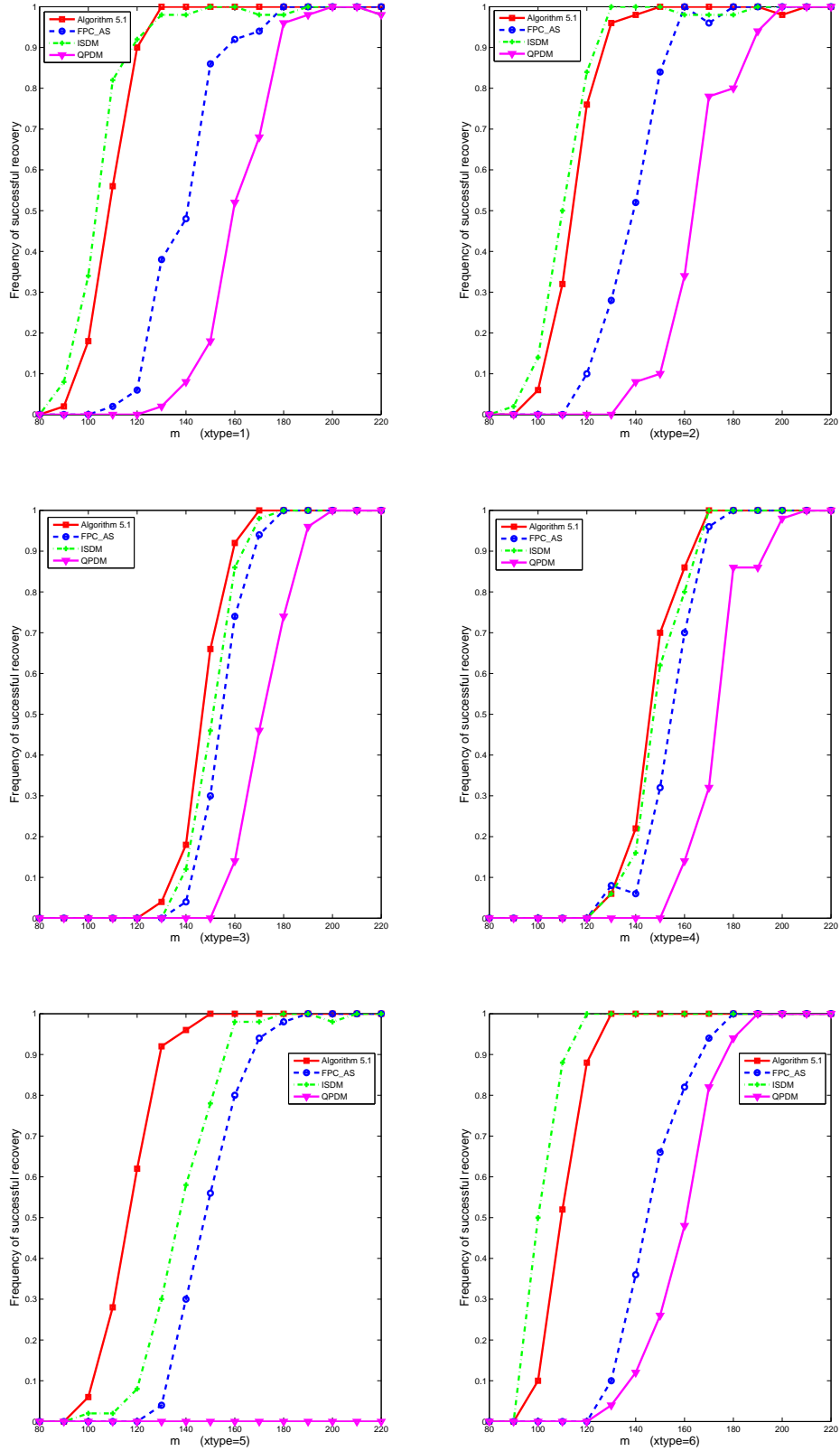


Figure 1: Frequency of successful recovery for four solvers ($Atype=1$)

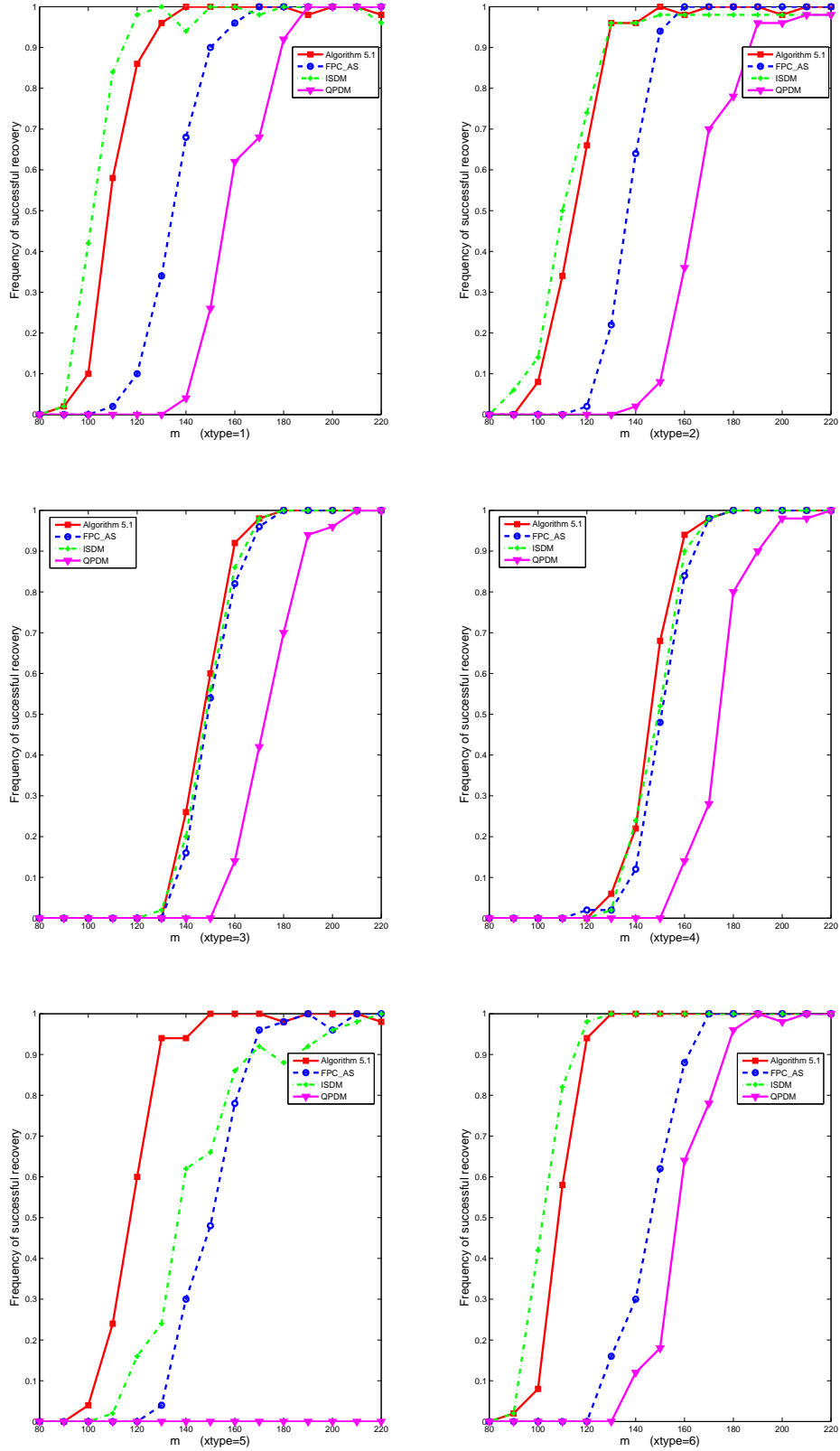


Figure 2: Frequency of successful recovery for four solvers ($Atype=2$)

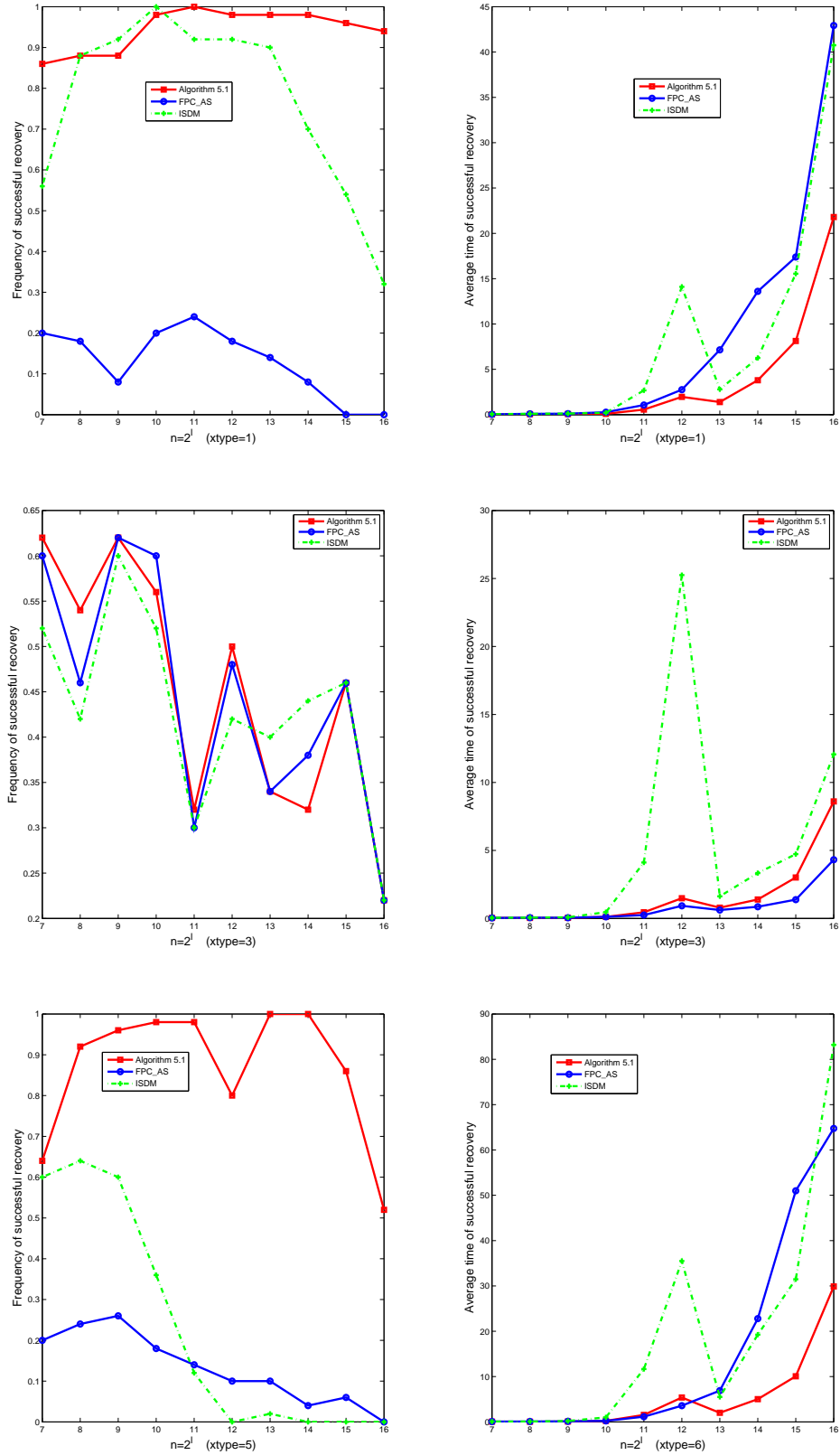


Figure 3: Frequency and time of successful recovery for three solvers (Atype= 5)

Figure 3 shows that for the signals of types 1 and 5, Algorithm 5.1 has much higher recoverability than **FPC_AS** and **ISDM** and requires the less recovery time; for the signals of type 3, the recoverability and the recovery time of three solvers are comparable.

From Figure 1-3, we conclude that for all types of matrices considered, Algorithm 5.1 has comparable even better recoverability than **ISDM** for the six types of signals above, and requires less computing time than **ISDM**; Algorithm 5.1 has better recoverability than **FPC_AS** and needs comparable even less computing time than **FPC_AS**; and **QPDM** has the worst recoverability for all types of signals. In view of this, we did not compare Algorithm 5.1 with **QPDM** in the subsequent numerical experiments.

5.3 Sparse signal recovery from noisy measurements

Since problems in practice are usually corrupted by noise, in this subsection we test the recoverability of Algorithm 5.1 on the same matrices and signals as in Subsection 5.2 but with Gaussian noise, and compare its performance with that of **FPC_AS** and **ISDM**. Specifically, we let $b = Ax^* + \theta\xi/\|\xi\|$, where ξ is a vector whose components are independently and identically distributed as $N(0, 1)$, and $\theta > 0$ is a given constant to denote the noise level. During the testing, we always set θ to 0.01, and chose the parameters of Algorithm 5.1 as in (28) and (29) except that $\epsilon = 1$, $\epsilon_1 = \frac{0.01\theta}{\max(1, \|b\|)}$,

$$\gamma = \begin{cases} 0.5 & \text{if } \|b\| \geq 10^2, \\ 0.8 & \text{otherwise,} \end{cases} \quad \hat{\gamma} = \begin{cases} 1 & \text{if } \|b\| \geq 10^2, \\ 10 & \text{otherwise.} \end{cases} \quad \text{and } j_{\max} = 5.$$

We first took the matrix of type 3 for example to test the influence of the number of measurements m on the recovery errors of three solvers for different types of signals. For each type of signals, we considered $n = 600$ and $K = 40$ and took the number of measurements $m \in \{120, 130, \dots, 240\}$. For each m , we generated 50 problems randomly and tested the recovery error of each solver. The curves in Figure 4 depict how the relative recovery error of three solvers vary with the number of measurements for different types of signals. From this figure, we see that for the signals of types 1, 2 and 6, Algorithm 5.1 and **ISDM** require less measurements to yield the desirable recovery error than **FPC_AS** does; for the signals of types 3 and 4, the three solvers are comparable in terms of recovery errors; and for the signals of type 5, **ISDM** yields a little better recovery error than Algorithm 5.1 and **FPC_AS**. After checking, we found that for the signals of type 5, the solutions yielded by **ISDM** have a large average residual; for example, when $m = 240$, the average residual attains 0.3, whereas the average residual yielded by Algorithm 5.1 and **FPC_AS** are less than 0.02. In other words, the solutions yielded by **ISDM** deviate much from the set $\{x \in \mathbb{R}^n \mid \|Ax - b\| \leq \delta\}$.

Finally, we took the matrix of type 4 for example to compare the recovery errors and the computing time of three solvers for the signals of a little higher dimension. Since Figure 4 shows that the three solvers have the similar performance for the signals

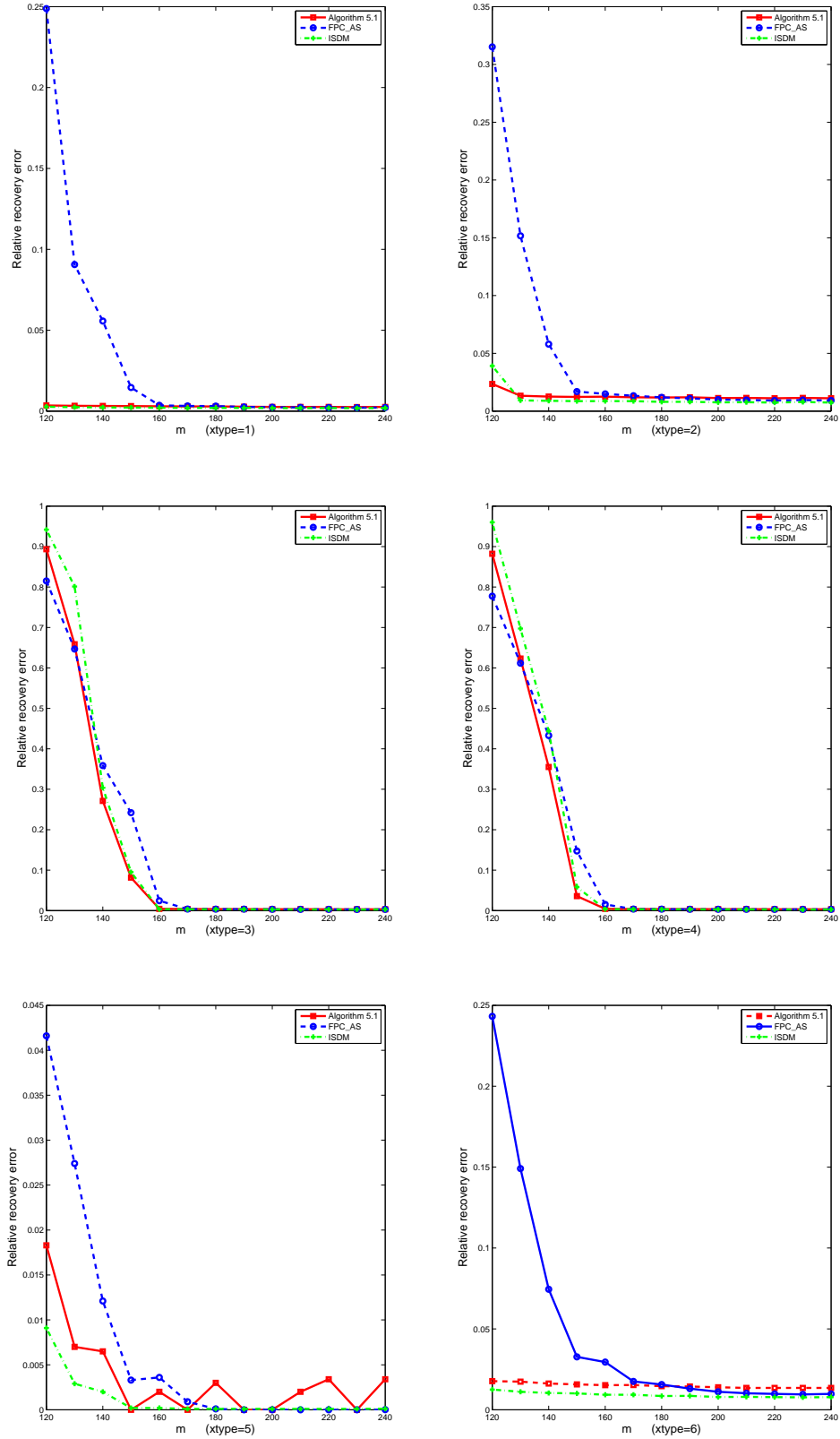


Figure 4: Relative recovery error of three solvers (Atype= 3)

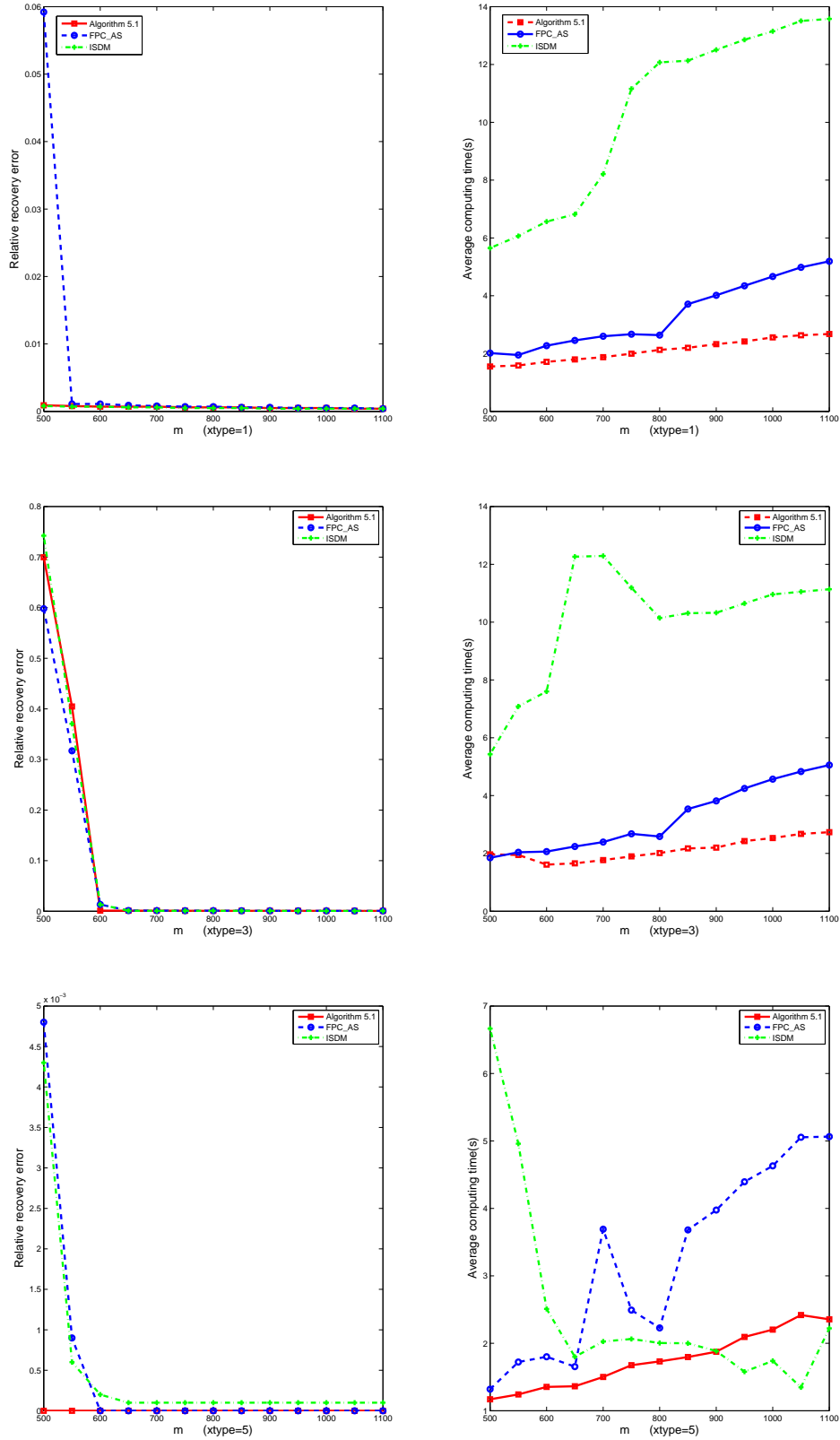


Figure 5: Relative recovery error and average computing time of three solvers ($A_{type}=4$)

of types 1, 2 and 6, and the similar performance for the signals of type 3 and 4, we compared the performance of three solvers only for the signals of types 1, 3 and 5. For each type of signals, we considered the dimension $n = 2^{11}$ and the number of nonzeros $K = 150$ and took the number of measurements $m \in \{500, 550, \dots, 1100\}$. For each m , we generated 50 problems randomly and tested the recovery error of each solver. The curves of Figure 5 depict how the recovery error and the computing time of three solvers vary with the number of measurements for different types of signals.

Figure 5 shows that for the signals of a little larger dimension, Algorithm 5.1 yields comparable recovery errors with **ISDM** and **FPC_AS** and requires less computing time than **ISDM** and **FPC_AS**. Together with Figure 4, we conclude that for the noisy signal recovery, Algorithm 5.1 is superior to **ISDM** and **FPC_AS** in terms of computing time, and comparable with **ISDM** and better than **FPC_AS** in terms of the recovery error.

5.4 Sparco collection

In this subsection we compare the performance of Algorithm 5.1 with that of **FPC_AS** and **ISDM** on **24** problems from the Sparco collection [3], for which the matrix A is stored implicitly. Table 3 reports their numerical results where, each column has the same meaning as in Table 2. When the true x^* is unknown, we mark **Relerr** as “—”.

From Table 3, we see that Algorithm 5.1 can solve those large-scale problems such as “srcsep1”, “srcsep2”, “srcsep3”, “angiogram” and “phantom2” with the desired feasibility, where “srcsep3” has the dimension $n = 196608$, and requires comparable computing time with **FPC_AS**, which is less than that required by **ISDM** for almost all the test problems. The solutions yielded by Algorithm 5.1 have the smallest zero-norm for almost all test problems, and have better feasibility than those given by **ISDM**. In particular, for those problems on which **FPC_AS** and **ISDM** fail (for example, “heavisgn”, “blkn-heavi” and “yinyang”), Algorithm 5.1 still yields the desirable results. Also, we find that for some problems (for example, “angiogram” and “phantom2”), the solutions yielded by **FPC_AS** have good feasibility, but their zero-norms are much larger than those of the solutions yielded by Algorithm 5.1 and **ISDM**.

From the numerical comparisons in Subsection 5.1-5.4, we conclude that Algorithm 5.1 is comparable even superior to **ISDM** in terms of recoverability, and the superiority of Algorithm 5.1 is more remarkable for those difficult problems from Sparco collection. The recoverability of Algorithm 5.1 and **ISDM** is higher than that of **FPC_AS**. In particular, Algorithm 5.1 requires less computing time than **ISDM**. The recoverability and recovery error of **QPDM** is much worse than that of the other three solvers.

Table 3: Numerical comparisons of three solvers on Sparco collection

No.	Problem	Solver	time(s)	Relerr	Res	nMat	nnzx
1	Heavisine	Algorithm 5.1	0.45	2.23e-12	2.23e-10	618	4
		FPC_AS	5.82	6.86e-1	1.54e+0	6891	1557
		ISDM	5.55	6.47e-1	2.17e-0	–	10
2	blocksig	Algorithm 5.1	0.14	2.53e-12	2.00e-10	539	71
		FPC_AS	0.03	1.07e-11	8.43e-10	9	71
		ISDM	0.03	7.94e-15	5.91e-13	–	71
3	cosspike	Algorithm 5.1	0.66	9.80e-13	9.95e-11	1546	115
		FPC_AS	0.13	1.10e-11	1.10e-9	221	115
		ISDM	5.99	4.25e-7	4.32e-5	–	115
4	zsinspike	Algorithm 5.1	0.89	1.05e-10	1.09e-8	1748	121
		FPC_AS	4.74	4.32e-11	4.82e-9	6669	121
		ISDM	5.13	3.68e-11	4.18e-9	–	121
5	gcosspike	Algorithm 5.1	1.31	1.06e-8	6.59e-7	1730	59
		FPC_AS	2.07	1.48e-11	8.42e-10	1123	59
		ISDM	8.81	4.34e-4	1.30e-2	–	61
6	p3poly	Algorithm 5.1	168.0	5.63e-9	2.06e-5	3851	166
		FPC_AS	23.4	5.85e-12	5.44e-11	1691	166
		ISDM	277.0	4.63e-3	9.05e-0	–	211
7	sgnspike	Algorithm 5.1	0.34	1.97e-11	4.33e-11	266	20
		FPC_AS	0.17	4.50e-10	9.36e-10	63	20
		ISDM	0.81	1.95e-14	3.59e-14	–	20
8	zsgnspike	Algorithm 5.1	1.37	5.04e-12	1.58e-11	609	20
		FPC_AS	33.3	3.57e-9	1.60e-8	6747	20
		ISDM	3.15	1.90e-14	5.02e-14	–	20
9	blkheavi	Algorithm 5.1	0.19	3.03e-8	8.92e-7	2137	12
		FPC_AS	0.14	2.45e-11	3.91e-10	789	12
		ISDM	5.69	7.30e+2	1.02e+4	–	102
10	blknheavi	Algorithm 5.1	0.45	3.29e-7	2.00e-6	2059	12
		FPC_AS	1.72	2.57e-2	1.40e-1	6797	344
		ISDM	2.17	6.05e-1	1.54e+0	–	25
11	gausspike	Algorithm 5.1	0.31	1.64e-9	1.46e-7	1671	32
		FPC_AS	0.14	3.16e-12	4.38e-11	181	32
		ISDM	0.99	2.89e-14	1.67e-12	–	32
12	srcsep1	Algorithm 5.1	186.0	–	6.81e-6	3431	21520
		FPC_AS	238.0	–	4.08e-5	6885	42676
		ISDM	388.0	–	3.81e-3	–	21644
13	srcsep2	Algorithm 5.1	380.0	–	1.92e-6	3628	21733
		FPC_AS	351.0	–	3.09e-4	6885	64478
		ISDM	601.0	–	2.01e-3	–	23258

14	srcsep3	Algorithm 5.1	409.0	–	1.83e-4	3934	110406
		FPC_AS	589.0	–	1.25e-7	7131	113438
		ISDM	316.0	–	3.78e-3	–	110599
15	phantom1	Algorithm 5.1	3.53	–	3.31e-6	2399	606
		FPC_AS	19.4	–	1.19e-5	7203	3989
		ISDM	9.08	–	2.22e-10	–	811
16	angiogram	Algorithm 5.1	1.59	–	6.96e-7	695	574
		FPC_AS	4.24	–	3.29e-9	485	9881
		ISDM	3.34	–	2.17e-13	–	574
17	phantom2	Algorithm 5.1	57.8	–	2.97e-7	2914	20962
		FPC_AS	44.4	–	1.04e-8	861	64599
		ISDM	149.0	–	5.93e-7	–	29313
18	smooth soccer	Algorithm 5.1	53.0	–	6.41e+1	5102	1831
		FPC_AS	97.8	–	5.18e+0	6875	3338
		ISDM	456.0	–	2.64e+3	–	2
19	soccer	Algorithm 5.1	35.8	–	1.45e-6	4427	701
		FPC_AS	25.7	–	4.61e-10	1769	701
		ISDM	247.0	–	1.00e+7	–	6
20	yinyang	Algorithm 5.1	8.70	–	5.65e-7	2710	771
		FPC_AS	35.4	–	7.99e-3	6563	3281
		ISDM	52.7	–	6.34e-4	–	886
21	blurrycam	Algorithm 5.1	167.0	–	4.43e-6	3992	62757
		FPC_AS	76.5	–	8.63e-7	2313	62757
		ISDM	525.0	–	2.51e-1	–	54829
22	blurspike	Algorithm 5.1	27.6	–	1.36e-6	3225	15592
		FPC_AS	11.7	–	4.86e-7	1863	15592
		ISDM	67.0	–	2.62e-3	–	15276
23	jitter	Algorithm 5.1	0.03	6.06e-10	3.02e-10	50	3
		FPC_AS	0.02	7.99e-10	3.84e-10	35	3
		ISDM	0.02	2.10e-14	9.69e-15	–	3
24	spiketrn	Algorithm 5.1	0.36	3.48e-9	1.27e-7	1851	12
		FPC_AS	1.45	3.81e-11	1.94e-9	4535	12
		ISDM	4.82	3.68e-0	1.40e+1	–	34

6 Conclusions

In this work we reformulated the zero-norm problem (1) as an equivalent MPEC, then established its exact penalty formulation (4). To the best of our knowledge, this novel result can not be obtained from the existing exact penalty results for MPECs. Motivated by the special structure of exact penalty problem, we proposed a decomposition method for dealing with the MPEC problem, and consequently the zero-norm problem. This method consists of finding the solution of a finite number of weighted l_1 -norm minimization problems, for which we propose an effective partial PPA algorithm for dealing with them. In particular, we show that this method can yield an optimal solution of the zero-norm problem under the null space condition used in [23]. Numerical comparisons show that the exact penalty decomposition method is significantly better than the quadratic penalty decomposition method [25], is comparable with **ISDM** in terms of recoverability [38] but requires less computing time, and has better recoverability than **FPC_AS** [39] and requires comparable computing time.

There are several research topics worthwhile to pursue; for example, one may consider to extend the results of this paper to rank minimization problems, design other effective convex relaxation methods for (1) based on its equivalent MPEC problem, and make numerical comparisons for the exact penalty decomposition method with the weighted l_1 -norm subproblems solved by different effective algorithms.

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